zero. Then by §2, the nine elements a_{11}, ..., a_{33} can be readily expressed rationally in terms of the M_2 and the square root of the second determinant (2).

Not all the minors A_{33}, A_{32}, A_{31} vanish, say $A_{32} \neq 0$. Having the minors $a_{11}a_{2i} - a_{21}a_{1i}$ and $a_{13}a_{2i} - a_{23}a_{1i}$, we get a_{2i} and a_{1i} since the determinant of their coefficients is $-A_{32}$. In this way we can determine a_{1i}, a_{2i}, a_{3i}, a_{i1}, a_{i2}, a_{i3} ($i=4$, ..., n). Of the elements previously found let $a_{13} \neq 0$, for example. We may now find a_{rs} ($r>3$, $s>3$) from $a_{13}a_{rs} - a_{1s}a_{r3}$.

4. As a further illustration, consider the case $m=3$. Of the determinants of order 4 built from the first four rows of D, suppose that

$$(3) \qquad \begin{vmatrix} 12 & 13 & 14 & 15 \\ \cdots & \cdots & \cdots & \cdots \\ 42 & 43 & 44 & 45 \end{vmatrix} \neq 0.$$

By §2, the sixteen elements of (3) are readily expressed as rational functions of the minors M_2 and the cube root of a rational function of the M_3. We next determine a_{1r}, a_{2r}, a_{3r} ($r=1, 6, 7, \ldots\ldots, n$). Consider the six M_3:

$$a_{3r}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{2i} & a_{2j} \end{vmatrix} - a_{2r}\begin{vmatrix} a_{1i} & a_{1j} \\ a_{3i} & a_{3j} \end{vmatrix} + a_{1r}\begin{vmatrix} a_{2i} & a_{2j} \\ a_{3i} & a_{3j} \end{vmatrix} \quad (i, j=2, 3, 4, 5;\ i<j).$$

From three of these M_3 we can find a_{3r}, $-a_{2r}$, a_{1r} if* the determinant of their coefficients is $\neq 0$. For $i, j=2, 3;\ 2, 4;\ 3, 4$, this determinant is

$$\begin{vmatrix} \begin{vmatrix} 12 & 13 \\ 22 & 23 \end{vmatrix} & \begin{vmatrix} 12 & 13 \\ 32 & 33 \end{vmatrix} & \begin{vmatrix} 22 & 23 \\ 32 & 33 \end{vmatrix} \\ \begin{vmatrix} 12 & 14 \\ 22 & 24 \end{vmatrix} & \begin{vmatrix} 12 & 14 \\ 32 & 34 \end{vmatrix} & \begin{vmatrix} 22 & 24 \\ 32 & 34 \end{vmatrix} \\ \begin{vmatrix} 13 & 14 \\ 23 & 24 \end{vmatrix} & \begin{vmatrix} 13 & 14 \\ 33 & 34 \end{vmatrix} & \begin{vmatrix} 23 & 24 \\ 33 & 34 \end{vmatrix} \end{vmatrix} = \begin{vmatrix} 12 & 13 & 14 \\ 22 & 23 & 24 \\ 32 & 33 & 34 \end{vmatrix}^2.$$

Indeed, by interchanging rows and columns in the first, we have a special case of (1). For $i, j=2, 3;\ 2, 5;\ 3, 5$, the determinant equals

$$\begin{vmatrix} 12 & 13 & 15 \\ 22 & 23 & 25 \\ 32 & 33 & 35 \end{vmatrix}^2,$$

etc. But not all minors of order 3 built from the first three rows of (3) vanish. Thus a_{1r}, a_{2r}, a_{3r} may be considered to be found. Of the minors from the first two rows of (3), let $a_{13}a_{24} - a_{14}a_{23} \neq 0$. We may thus get a_{4r} from

*This determinant vanishes if $i, j=2, 3;\ 2, 4;\ 2, 5$.

$$\begin{vmatrix} 1r & 13 & 14 \\ 2r & 23 & 24 \\ 4r & 43 & 44 \end{vmatrix}.$$

Similarly, we can find every a_{s2}, a_{s3}, a_{s4}, a_{s5}. Finally, from

$$\begin{vmatrix} 1l & 13 & 14 \\ 2l & 23 & 24 \\ kl & k3 & k4 \end{vmatrix}$$

we determine a_{kl} (k any $\neq 1, 2, 3, 4$; l any $\neq 2, 3, 4, 5$).

5. The following method of solution for the important case $m=2$ offers greater symmetry than that of §3. We show that, if $t, i_1, i_2, \ldots\ldots, i_{n-2}$ are any positive integers $\leqq n$, $a_{ti_1}, a_{ti_2}, \ldots\ldots, a_{ti_{n-2}}D$ equals a certain determinant of order $n-1$ whose elements are all minors of degree 2 of $D \equiv |a_{ij}|$. Let j_1 be any integer $\neq i_1$, of the set $1, \ldots\ldots, n$; let j_2 be any one $\neq i_2$ or j_1; j_3 any $\neq i_3, j_1, j_2$; $\ldots\ldots$; j_{n-2} any $\neq i_{n-2}, j_1, j_2, \ldots\ldots, j_{n-1}$; j_{n-1} and j_n the remaining two. Set $i_{n-1} = j_n$. Call C_i the ith column in D. Subtract $a_{tj_1}C_{i_1}$ from $a_{ti_1}C_{j_1}$, $a_{tj_2}C_{i_2}$ from $a_{ti_2}C_{j_2}$, $\ldots\ldots$, $a_{tj_{n-1}}C_{i_{n-1}}$ from $a_{ti_{n-1}}C_{j_{n-1}}$. Hence $a_{ti_1}\ldots\ldots a_{ti_{n-1}}D$ equals a determinant the elements of whose j_nth column are the same as in D, while the element lying in the j_sth column ($s<n$) and rth row is

$$a_{ti_s}a_{rj_s} - a_{tj_s}a_{ri_s} \equiv T_{rj_s}.$$

Hence in the tth row all the elements are zero except that in the j_nth column. Thus the new determinant is a_{tj_n} times its co-factor. Hence

$$(4) \quad a_{ti_1}\ldots\ldots a_{ti_{n-2}}D = (-1)^{t+j_n}\,|\,T_{rj_s}\,| \quad (s=1, \ldots\ldots, n-1;\ r=1, \ldots\ldots, t-1, t+1, \ldots\ldots, n),$$

the sequence of columns being that of the order of magnitude of $j_1, \ldots\ldots, j_{n-1}$. By interchanging the rows and columns of D we obtain similarly

$$(5) \quad a_{i_1t}\ldots\ldots a_{i_{n-2}t}D = \text{rational functions of minors of degree 2.}$$

Giving fixed values to $i_2, \ldots\ldots, i_{n-2}$, and varying i_1, we conclude that the ratio of any two elements a of the same row (or column) is a known rational function of the minors of degree 2. Thus if $a_{t\sigma} \neq 0$, $a_{tl} = R_l = a_{t\sigma}$, $a_{k\sigma} = R_k' a_{t\sigma}$, for every l, k. To prove that $a^2{}_{t\sigma}$ is a rational function of the minors of degree 2, consider any non-vanishing minor D_4 of degree 4 of D. By Laplace's expansion, D_4 is a rational function of the minors of degree 2. Applying (1) for $n=4$, $i_1 = i_2 = \sigma$, we conclude the same for $a^2{}_{t\sigma}D_4$. Hence $a^2{}_{t\sigma} = R$, where R is a rational function of the minors of degree 2. Now the minor

$$\begin{vmatrix} a_{t\sigma} & a_{tl} \\ a_{k\sigma} & a_{kl} \end{vmatrix} = a_{t\sigma}a_{kl} - R_l R'_k a^2{}_{t\sigma} = R(a_{kl}/a_{t\sigma} - R_l R'_k).$$

Hence $a_{ki} \div a_{t\sigma}$ is a rational function of the minors of degree 2. *Every element a is a rational function of $\sqrt{R}$ and the minors.*

6. In conclusion I establish a general theorem on determinants which includes (4) as one special case, and Studnicka's theorem (cited in §1) as another very special case. Let $r_1, r_2, \ldots, r_{m-1}$ be any distinct integers of the set $1, \ldots, n$; likewise for $i_{11}, i_{12}, \ldots, i_{1\,m-1}, j_1$. Let r be any integers $\geqq n$. In

$$M_{rj_1} \equiv \begin{vmatrix} a_{ri_{11}} & a_{ri_{12}} \cdots a_{ri_{1m-1}} & a_{rj_1} \\ a_{r_1 i_{11}} & a_{r_1 i_{12}} \cdots a_{r_1 i_{1m-1}} & a_{r_1 j_1} \\ \cdots & \cdots & \cdots \\ a_{r_{m-1} i_{11}} & a_{r_{m-1} i_{12}} \cdots a_{r_{m-1} i_{1m-1}} & a_{r_{m-1} j_1} \end{vmatrix}$$

call $A_{ri_{11}}, \ldots, A_{rj_1}$ the co-factors of the elements of the first row. Call C_i the ith column in $D \equiv |\, a_{ij} \,|$. Multiply $C_{i_{11}}$ by $A_{ri_{11}}$, $C_{i_{12}}$ by $A_{ri_{12}}, \ldots, C_{i_{1m-1}}$ by $A_{ri_{1m-1}}$, and add the products to $C_{j_1} \times A_{rj_1}$. Hence $A_{rj_1} D$ equals a determinant whose elements outside column C_{j_1} are the same as in D, while the element lying in the j_1th column and rth row is M_{rj_1}, whence those in the rows $r_1, \ldots, r_{m-1}$ are zero. Next, let $i_{21}, i_{22}, \ldots, i_{2m-1, j_2}$ be any distinct integers other than j_1 of the set $1, \ldots, n$. In

$$\mathrm{M}_{rj_2} \equiv \begin{vmatrix} a_{ri_{21}} & a_{ri_{22}} & \cdots a_{ri_{2m-1}} & a_{rj_2} \\ a_{r_1 i_{21}} & a_{r_1 i_{22}} & \cdots a_{r_1 i_{2m-1}} & a_{r_1 j_2} \\ \cdots & \cdots & \cdots & \cdots \\ a_{r_{m-1} i_{21}} & a_{r_{m-1} i_{22}} & \cdots a_{r_{m-1} i_{2m-1}} & a_{r_{m-1} j_2} \end{vmatrix}$$

call $A_{ri_{21}}, \ldots, A_{rj_2}$ the co-factors of the elements of the first row. Add $C_{i_{21}} A_{ri_{21}}, \ldots, C_{i_{2m-1}} A_{ri_{2m-1}}$ to $C_{j_2} A_{rj_2}$. Hence $A_{rj_1} A_{rj_2} D$ equals a determinant whose elements outside columns C_{j_1}, C_{j_2} are the same as in D, while the elements in columns j_1 and j_2 and rth row are M_{rj_1} and M_{rj_2}, respectively. For the third step of the argument, $i_{31}, \ldots, i_{3m-1, j_3}$ are any distinct integers other than j_1, j_2 of the set $1, \ldots, n$. For the $(n-m+1)$th step, $i_{n-m+1}, \ldots, i_{n-m+1\, m-1}$, j_{n-m+1} are any distinct integers other than $j_1, j_2, \ldots, j_{n-m}$ of the set $1, \ldots, n$. After this final step, $A_{rj_1} A_{rj_2}, \ldots, A_{rj_{n-m+1}} D$ equals a determinant whose elements outside columns $C_{j_1}, \ldots, C_{j_{n-m+1}}$ are the same as in D, while the elements in these columns and the rth row are $M_{rj_1}, \ldots, M_{rj_{n-m+1}}$, respectively. In particular, the latter elements are obviously zero for $r = r_1, \ldots, r_{m-1}$. By Laplace's development the determinant is, apart from sign, $A_{rj_{n-m+1}} \times$ co-factor. The sign is determined by the fact that our co-factor is gotten by deleting the rows $r_1, \ldots, r_{m-1}$ and the columns $i_{n-m+1\,1}, \ldots, i_{n-m+1\, m-1}$. Thus

$$A_{rj_1} A_{rj_2}, \ldots, A_{rj_{n-m}} D = \pm \,|\, M_{rj_s} \,| \tag{6}$$

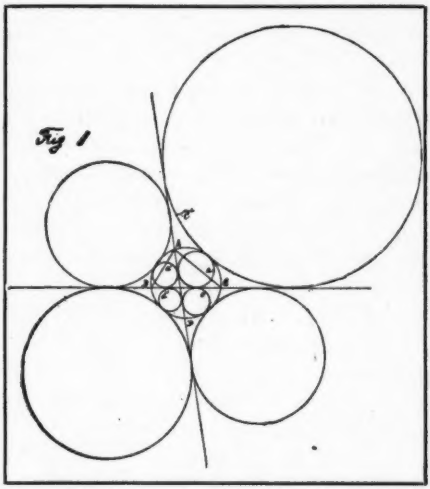
($s = 1, \ldots, n-m+1$; $r = 1, \ldots, n$; $r \neq r_1, \ldots, r_{m-1}$), where the diagonal term of the determinant A_{rj_s} is $a_{r_1 i_{s_1}} a_{r_2 i_{s_2}}, \ldots, a_{r_{m-1}\, i_{sm-1}}$.

THE UNIVERSITY OF CHICAGO, *November, 1905.*

A NEW GEOMETRICAL PROPOSITION.

By Y. SAWAYAMA, Instructor in The Central Military School for Boys, Tokyo, Japan.

General enunciation.—Describe eight circles, each of which touches a circle and any two secants of it; next construct a triangle by joining any three of the points of intersection of the latter circle and the two secants; then the chords of contact and the line of centers of two of the eight circles taken appropriately in pairs are concurrent, the point of concurrency being equidistant from the three sides of the triangle.

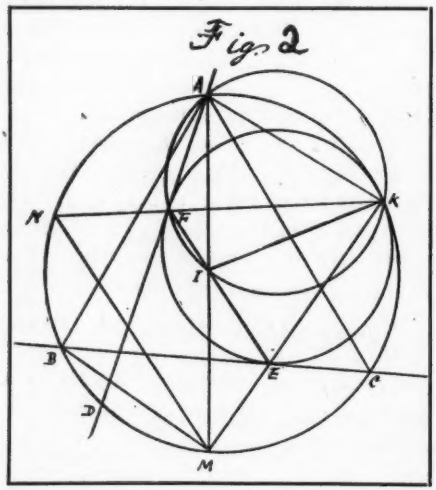


Particular enunciation. — Call the circle, ABC, the eight circles a, a', b, b', c, c', d, d', and the two secants AD, BC (Fig. 1). Then

1. The line of centers of the two circles a and a', and their two chords of contact pass through the center of the inscribed circle of the triangle ABC.

2. The line of centers of the two circles b and b', and their two chords of contact pass through the center of the circle escribed to the side BC of the triangle ABC.

3. The line of centers of the two circles c and c', and their two chords of contact pass through the center of the circle escribed to the side CA of the triangle ABC.



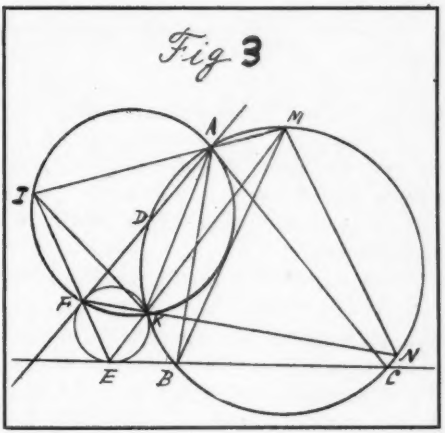
4. The line of centers of the two circles d and d', and their two chords of contact pass through the center of the circle escribed to the side AB of the triangle ABC.

Demonstration.—Let K, E, and F be the three points of contact where any one of the eight circles touches the circle ABC and the two secants BC, AD (Figs. 2 and 3). Then as is well known, we have two properties:

1. The line KE passes through the middle point M of the arc BC, so also the line KF through the middle point N of the arc AD.

2. The two straight lines EF and MN are parallel.

From the last property we have $\angle EFK = \angle MNK$, and as the four points M, N, A, and K lie on the circle ABC, $\angle MNK = \angle MAK$ (Fig. 2), or =its supplementary angle (Fig. 3).



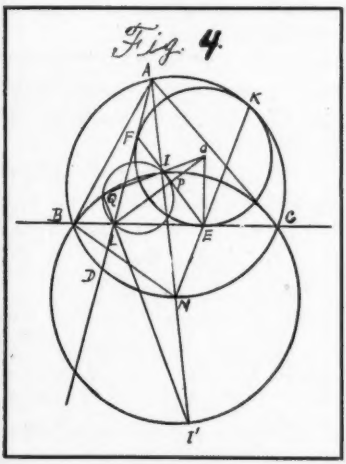
$\therefore \angle EFK = \angle MAK$ or=its supplementary angle. Hence the four points A, K, F, and I (which is the intersection of EF and AM) are concyclic.

$\therefore \angle AIK = \angle AFK$ (Figs. 2 and 3), or=its supplementary angle (figure omitted). Now the line AF touches the circle EFK, hence $\angle AFK$ or its supplementary angle=FEK. $\therefore \angle AIK = \angle FEK$.

$\therefore \triangle MKI$ and $\triangle MIE$ are mutually equiangular, from which we can deduce that the circle passing through the three points I, E, and K touches the line MI, hence $MI^2 = MK.ME$. On the other hand, since arcMB=arcMC, we have $\angle MKB = \angle MBE$ (in the figures we omit the line BK).

$\therefore$ The circle passing through the three points B, E, and K touches the line MB, hence we have $MK.ME = MB^2$, or $MI^2 = MB^2$, $MI = MB$.

This shows that I is equidistant from the three sides of the triangle ABC.



Next, call O the center of the circle EFK, L the intersection of BC and AD, P the intersection of EF and OL, Q another intersection of OI and the circle BIC, and draw the straight line passing through Q and L (Fig. 4).

Then the square of the tangent drawn from the center of the circle BIC to the circle EFK is equal to the rectangle $MK.ME$, or the square of the radius of the circle BIC.

This last fact shows that the two circles EFK and BIC are orthogonal, and the radius OE of the circle EFK is equal to the tangent drawn from the point O to the circle BIC, and that $OE^2 = OQ.OI$.

Again, in the right-angled triangle OLE, $OE^2 = OL.OP$. $\therefore OQ.OI = OL.OP$.

$\therefore$ The four points I, P, L, and Q are concyclic.

Noting the angle IPL is a right angle and the $\angle IQL$ and $\angle IPL$ are equal or supplementary, we have $\angle IQL$=right angle.

$\therefore$ The line QL passes through a point I' which is equidistant from the three sides of the triangle ABC (I' is, in fact, the other extremity of the diameter passing through I of the circle BIC). And the line OI is perpendicular to a fixed line $I'L$ passing through the fixed point Q on it.

Similarly, the center of another circle whose chord of contact passes through the point I, and the point I are on the straight line which is perpendicular to $I'L$ and passes through the fixed point Q on it. Therefore the centers of the two circles whose chords of contact pass through I and the point I are collinear.

Corollary I. In the preceding figures, the point of intersection (other than I) of the circle BIC and the line EF is one of the points which are equidistant from the three sides of the triangle DBC.

One of the two intersections of the line EF and the circle whose center is N and whose radius is NA is equidistant from the three sides of the triangle BAD, and the other from the three sides of the triangle CAD.

Corollary II. In Fig. 1 the points of contact of the eight circles with either of the two secants are in involution, the center of which is the intersection L of the two secants, and then its constant is equal to the power of the point L with respect to the circle ABC.

Demonstration. Take L, the intersection of two secants AD and BC as the center of inversion, and the power of L with respect to the circle ABC for the constant of inversion. In the figures 2, and 3, call E', F', and K', the inverses of the three points E, F, and K, respectively. Then as the two secants EC and AD, and the circle ABC are their own inverses, the circle $E'F'K'$ which is the inverse of the circle EFK, touches the secants BC and AD and the circle ABC.

Corollary III. When AD and BC are perpendicular, the centers of the four circles a, b, a', b' are concyclic; so also those of a, b, c, d; those of a', b', c', d', and those of c, d, c', d'.

DEPARTMENTS.

*DISCUSSION.

THE TANGENT NORMALS TO A LIMACON.

By F. H. SAFFORD, Ph. D., The University of Pennsylvania.

This problem was suggested by Dr. Glenn at the end of the solution of Problem 254, August-September, 1905.

Let the required line be $2ax+2by+c=0$, which is to be both tangent and normal to

$$(x^2+y^2+cx)^2=\frac{c^2}{e^2}(x^2+y^2).$$

Two special cases may be disposed of first: one is that of tangent lines parallel to the Y-axis, the other the case of tangents passing through the origin.

The latter is excluded from the general case by the particular value chosen for the constant term in the straight line, but it may be shown that the only line which satisfies the problem in this case is $y=0$ for $e=1$, *i. e.* the axis of the cardioid.

The former case leads to a solution when $e=\frac{3+\sqrt{5}}{2}$, the point of tangency being at the extremity of the loop, while the line is normal at both of the remaining intersections. The general solution now proceeds under the assumptions $b\neq 0$, $c\neq 0$, and is entirely analogous to the solution previously given, p. 157, for the case in which $e=1$.

Transform the line and limaçon by the inversion

$$x=\frac{cx'}{x'^2+y'^2},\quad y=\frac{cy'}{x'^2+y'^2},$$

obtaining $x^2+y^2+2ax+2by=0$, $x^2(1-e^2)+y^2-2e^2x-e^2=0$.

Since angles are unchanged by inversion the problem is now to determine a and b so that the circle shall be both tangent and normal to the conic. The two curves have a point of orthogonal intersection when $x=-1$ or $1-2a$, but the first value leads to $x^2+y^2=0$, which is excluded. When $x=1-2a$, in which $a\neq 1$ for the same reason, then y and b may be found in terms of a as a parameter.

It will simplify later work to write $a=1+h$ $(h\neq 0)$, $e^2=1+d$, so that the point in question is

*The department *Discussion* will be devoted primarily to generalizations and extensions of problems solved in our columns. Some important researches have resulted from such generalizations in the past and it is believed that a department devoted to such investigations is desirable. G.

$$x=-1-2h,$$

$$y=\frac{-2h^2d+3h+1}{b},$$

$$b=\frac{(-2h^2d+3h+1)^2}{4h^2d-4h-1} \quad (b\neq\infty).$$

Eliminating y from the equations of circle and conic gives a biquadratic whose roots are of course the abscissae of intersection points, one of which has been found above. The depressed equation should have a double root to give the desired tangency. The discriminant $\triangle$ of this cubic must vanish, thus giving an equation for the determination of h and thence of a and b in terms of e. For this cubic the coefficients are

$$a_0=d+1,\ 3a_1=h(-2d+2)+3d+7,$$

$$3a_2=\frac{h^3(16d^2)+h^2(-12d^2-56d)+h(8d+44)+3d+11}{N},$$

$$a_3=\frac{h^3(8d^2)+h^2(-4d^2-24d)+h(2d+18)+d+5}{N},$$

in which $N=1-4h^2d+4h$, also

$$\triangle\equiv 4(a_0a_2-a_1^2)(a_1a_3-a_2^2)-(a_0a_3-a_1a_2)^2=0.$$

This form of $\triangle$ leads to some saving of labor over that of the expanded form.

The following are given as aids in comparison of results:

$$a_0a_2-a_1^2=\frac{4}{9N}[h^2(4d^3-8d^2+4d)+h^3(-8d^2+36d-4)+h^2(2d^2+25d-29)$$
$$+h(d-23)-4]$$

$$-(a_1a_3-a_2^2)=\frac{4}{9N^2}[h^6(16d^4+48d^3)+h^5(-112d^3-192d^2)$$
$$+h^4(4d^3+236d^2+252d)+h^3(20d^2-188d-108)$$
$$+h^2(-2d^2-57d+49)+h(-d+35)+4]$$

$$a_0a_3-a_1a_2=\frac{8}{9N}[h^4(4d^3-4d^2)+h^3(-16d^2+14d)+h^2(2d^2+31d-11)$$
$$+h(d-21)-4].$$

The final equation giving h, is

$$16h^8d^4(d-1)^2-16h^7d^3(4d^2-17d+5)-4h^6d^2(4d^3-31d^2+219d-37)$$

$$+4h^5d(10d^3-80d^2+337d-30)+h^4(4d^4-40d^3+682d^2-997d+36)$$

$$-h^3(4d^3-92d^2+379d-285)+h^2(2d^2-147d+241)-h(9d-71)+7=0.$$

When $d=0$, corresponding to the cardioid, only one of the four roots is available, two of the others being excluded since $b\neq 0$ and the fourth one because $b\neq\infty$, already included in the second special case.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

245. Proposed by S. I. JONES, A. B., Gunter Bible College, Gunter, Texas.

The shell of a hollow iron ball is 4 inches thick, and contains $\frac{1}{5}$ of the number of cubic inches in the whole ball. Find the diameter of the ball.

I. Solution by S. A. COREY, Hiteman, Iowa.

Let r be the radius of the ball; $(r-4)$ will then be the radius of the hollow sphere enclosed by the shell. As the volumes of spheres are proportional to the cubes of their radii, the conditions of the problem require that

$r^3-(r-4)^3=\frac{1}{5}r^3$, or $\frac{4}{5}r^3=(r-4)^3$, whence, $r=\frac{4}{1-\sqrt[3]{\frac{4}{5}}}=55.79$ inches, nearly.

II. Solution by M. R. BECK, Cleveland High School, Ohio.

Let r be the radius of the ball, then $\frac{4}{3}\pi r^3=\frac{4}{15}\pi r^3+\frac{4}{3}\pi(r-4)^3$.....(1).

From (1) we have $r^3-60r^2+240r-320=0$.....(2).

Substitute $r=x+\frac{320}{x}+20$ in (2), $x^3+\frac{32{,}768{,}000}{x^3}-11520=0$.....(3).

Solving the quadratic (3), $x=\sqrt[3]{(6400)}$ or $\sqrt[3]{(5120)}$.

Either root makes $r=55.8016$, and the diameter is 111.6032 inches.

Also solved by P. S. Berg, G. W. Greenwood, A. H. Holmes, L. E. Newcomb, D. B. Northrup, J. Scheffer, J. E. Sanders, and G. B. M. Zerr.

AVERAGE AND PROBABILITY.

169. Proposed by HENRY HEATON, Atlantic, Iowa.

*What is the average length of all straight lines that can be drawn within a given square in every possible direction and every possible length from every point of the square; if all the lines are equally distributed about the starting point and equally distributed as to length.

*The problem as restated above is somewhat different from the one solved in our columns last month. As the above conveys the original meaning of the Proposer it is published as a third solution.

III. Solution by the PROPOSER.

Let $ABCD$ be the given square whose side$=a$, and MN one of the straight lines within the square. Draw AE equal and parallel to MN, EF parallel to AB cutting BC in F, and EG parallel to AD cutting CD in G. Put $MN=AE=x$ and $\angle EAD=\theta$. If x and θ were fixed the number of lines of the length a would equal the number of points in the rectangle $EFCG$ whose area is $a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta$. Then if all lines are supposed to be equally distributed about the point M the required average is

$$A_1=\frac{\int_0^{\frac{1}{2}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]x\,dx}{\int_0^{\frac{1}{2}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]\,dx}$$

$$=\frac{\frac{a^4}{12}\int_0^{\frac{1}{2}\pi}(2-\tan\theta)\sec^2\theta\,d\theta}{\frac{a^3}{6}\int_0^{\frac{1}{2}\pi}(3-\tan\theta)\sec\theta\,d\theta}=\frac{3a}{4[\log(1+\sqrt{2})^3+1-\sqrt{2})}.$$

If the lines are supposed to be so distributed as to join every possible pair of points in the square, the required average is

$$A_2=\frac{\int_0^{\frac{1}{2}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]x^2\,dx}{\int_0^{\frac{1}{2}\pi}d\theta\int_0^{a\sec\theta}[a^2-ax(\sin\theta+\cos\theta)+x^2\sin\theta\cos\theta]x\,dx}$$

$$=\frac{\frac{a^5}{60}\int_0^{\frac{1}{2}\pi}(5-3\tan\theta)\sec^3\theta d\theta}{\frac{a^4}{12}\int_0^{\frac{1}{2}\pi}(2-\tan\theta)\sec^2\theta d\theta}=\frac{a}{15}[2+\sqrt{2}+5\log(1+\sqrt{2})].$$

172. Proposed by J. EDWARD SANDERS.

What is the average length of all straight lines that can be drawn within a given triangle?

*Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Let a, b, c be the sides; A, B, C the angles of the triangle; u, v variable lengths from A on b, c; w, x variable lengths from B on a, c; y, z variable lengths from C on a, b. Also let $l_1=\sqrt{(u^2+v^2-2uv\cos A)}$, $l_2=\sqrt{(w^2+x^2-2wx\cos B)}$, $l_3=\sqrt{(y^2+z^2-2yz\cos C)}$, $M=$average length. Then

*Another solution of this problem will be published next month.

$$M=\frac{\int_0^b\int_0^c\int_0^{l_1} l\,du\,dv\,dl+\int_0^a\int_0^c\int_0^{l_2} l\,dw\,dx\,dl+\int_0^a\int_0^b\int_0^{l_3} l\,dy\,dz\,dl}{\int_0^b\int_0^c\int_0^{l_1} du\,dv\,dl+\int_0^a\int_0^c\int_0^{l_2} dw\,dx\,dl+\int_0^a\int_0^b\int_0^{l_3} dy\,dz\,dl}$$

$$=N/D.$$

$$\int_0^b\int_0^c\int_0^{l_1} l\,du\,dv\,dl=\tfrac{1}{2}\int_0^b\int_0^c (n^2+v^2-2uv\cos A)\,du\,dv$$

$$=\tfrac{1}{6}\int_0^b (3u^2c+c^3-3uc^2\cos A)\,du=\frac{bc}{12}(2b^2+2c^2-3bc\cos A).$$

$$\therefore N=\frac{2abc(a+b+c)+a^2b^2\cos C+a^2c^2\cos B+b^2c^2\cos A}{12}.$$

This follows, because $\frac{bc}{12}(2b^2+2c^2-3bc\cos A)=\frac{bc}{12}(2a^2+bc\cos A)$.

$$\int_0^b\int_0^c\int_0^{l_1} du\,dv\,dl=\int_0^k\int_0^c \sqrt{(u^2+v^2-2uv\cos A)}\,du\,dv$$

$$=\tfrac{1}{2}\int_0^b \Big[u^2\cos A+(c-u\cos A)\sqrt{(u^2+c^2-2uc\cos A)}$$

$$+u^2\sin^2 A\,\log\Big(\frac{c-u\cos A+\sqrt{(u^2+c^2-2uc\cos A)}}{n(1-\cos A)}\Big)\Big]du$$

$$=\tfrac{1}{6}(b^3+c^3-a^3)\cos A+\tfrac{1}{3}abc\sin^2 A+\tfrac{1}{6}b^3\sin^2 A\,\log(\cot\tfrac{1}{2}A\,\cot\tfrac{1}{2}B)$$

$$+\tfrac{1}{6}c^3\sin^2 A\,\log(\cot\tfrac{1}{2}A\,\cot\tfrac{1}{2}C).$$

$$\therefore D=\tfrac{1}{6}[(b^3+c^3-a^3)\cos A+(a^3+c^3-b^3)\cos B+(a^3+b^3-c^3)\cos C+2abc(\sin^2 A$$

$$+\sin^2 B+\sin^2 C)+b^3\sin^2 A\,\log(\cot\tfrac{1}{2}A\,\cot\tfrac{1}{2}B)+a^3\sin^2 B\,\log(\cot\tfrac{1}{2}A\,\cot\tfrac{1}{2}B)$$

$$+(c^3\sin^2 A+a^3\sin^2 C)\log(\cot\tfrac{1}{2}A\,\cot\tfrac{1}{2}C)$$

$$+(b^3\sin^2 C+c^3\sin^2 B)\log(\cot\tfrac{1}{2}B\,\cot\tfrac{1}{2}C)].$$

If the line is terminated by two of the sides

$$M_1=\frac{\int_0^b\int_0^c l_1\,du\,dv+\int_0^a\int_0^c l_2\,dw\,dx+\int_0^a\int_0^b l_3\,dy\,dz}{\int_0^b\int_0^c du\,dv+\int_0^a\int_0^c dw\,dx+\int_0^a\int_0^b dy\,dz}=\frac{D}{bc+ac+ab}.$$

Corollary. If $a=b=c$, $N=\frac{5}{8}a^4$, $D=a^3(1+\log 3)$.

$$\therefore M=\frac{5a}{8(1+\log 3)}. \quad M_1=\tfrac{1}{8}a(1+\log 3).$$

CALCULUS.

207. Proposed by F. P. MATZ, Sc. D., Ph. D.

If K represents the complete elliptic integral of the first kind, prove that

$$\int_0^1 \frac{K d\kappa}{1+\kappa}=\tfrac{1}{4}\pi^2.$$

Solution by S. A. COREY, Hiteman, Iowa.

As $K=\int_0^1 \frac{dx}{\sqrt{[(1-x^2)(1-\kappa^2x^2)]}}$, the definite integral may be written

$$\int_0^1\int_0^1 \frac{dx\,d\kappa}{(1+\kappa)\sqrt{[(1-x^2)(1-\kappa^2x^2)]}},$$

$$\text{or}\int_0^1 \frac{1}{(1-x^2)^{\frac{1}{2}}}\left[\int_0^1 \frac{d\kappa}{(1+\kappa)(1-\kappa^2x^2)^{\frac{1}{2}}}\right]dx \ldots\ldots (1).$$

If $r=\kappa+1$, $\int_0^1 \frac{d\kappa}{(1+\kappa)\sqrt{(1-\kappa^2x^2)}}=\int_1^2 \frac{dr}{r\sqrt{X}}=$

$$-\left[\frac{1}{\sqrt{a}}\log\left(\frac{\sqrt{X}+\sqrt{a}}{x}+\frac{b}{2\sqrt{a}}\right)\right]_1^2=-\frac{1}{\sqrt{a}}\log\left(\frac{1}{1+\sqrt{a}}\right)=\frac{1}{\sqrt{a}}\log(1+\sqrt{a}),$$

in which $a=(1+x^2)$, $b=2x^2$, and $X=(1-x^2)+2x^2r-x^2r^2$.

The definite integral (1), after substituting, becomes,

$$\int_0^1 \frac{\log[1+\sqrt{(1-x^2)}]}{(1-x^2)}dx, \text{ or, if } x=\frac{1-z^2}{1+z^2}, \quad \int_0^1 \frac{1}{z}\cdot\log\left(\frac{(1+z)^2}{1+z^2}\right)dz$$

$$=2\int_0^1 \frac{\log(1+z)}{z}dz-\int_0^1 \frac{\log(1+z^2)}{z}dz=2.\frac{\pi^2}{6}-\frac{\pi^2}{2.6}=\frac{\pi^4}{4}.$$

208. Proposed by F. P. MATZ, Sc. D., Ph. D.

Solve the differential equation

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0.$$

Solution by A. H. HOLMES, Brunswick, Me.

$$(a^2+x^2)\frac{d^2y}{dx^2}+2x\frac{dy}{dx}=0=\frac{d}{dx}\left[(a^2+x^2)\frac{dy}{dx}\right].$$

$$\therefore (a^2+x^2)\frac{dy}{dx}=C;\ dy=\frac{Cdx}{a^2+x^2};\ y=\frac{C}{a}\tan^{-1}\frac{x}{a}+C'.$$

Also solved by S. A. Corey, M. E. Graber, G.W. Greenwood, J. O. Mahoney, E. L. Rich, J. Scheffer, J. E. Sanders, and G. B. M. Zerr.

GEOMETRY.

268. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

Find, without the aid of trigonometry, the side of an inscribed regular polygon of $2n$ sides, if the side of an inscribed regular polygon of n sides is 16 feet. [Wentworth's *Plane Geometry*, Revised Edition, problem 512, page 244.]

Solution by M. R. BECK, Cleveland, Ohio.

Let AB be the chord=16. From the center O draw the radius $r=OC$, perpendicular to AB at D. Draw the chord BC=the side of an inscribed regular polygon of $2n$ sides.

In $\triangle OBD$, $r^2=(r-DC)^2+64$, and therefore $DC=r-\sqrt{(r^2-64)}$. Also in $\triangle CDB$, $BC^2=DC^2+64$. Substituting, $BC=\sqrt{\{2r[r-\sqrt{(r^2-64)}]\}}$.

Also solved by P. S. Berg.

269. Proposed by J. SCHEFFER, A. M.

Find the area of a segment, if the chord of the segment is 10 feet, and the radius of the circle is 16 feet.

Solution by P. S. BERG, Larimore, N. D.

Since $16-\sqrt{(16^2-5^2)}=.8014$, height of segment, then

$\frac{(.8014)^3}{2\times 10}+\frac{2}{3}\times .8014\times 10=5.36$ square feet, area of segment.

Also solved by G. W. Greenwood, M. E. Graber, A. H. Holmes, D. B. Northrup, J. Edward Sanders, and G. B. M. Zerr.

270. Proposed by F. R. HONEY, Ph. B., Hartford, Conn.

What portion of the heavens is always invisible to an observer whose latitude is given?

Solution by J. SCHEFFER, A. M., Hagerstown, Md.

That portion of the heavens which is never visible to an observer in a certain latitude is that cut off by the circle of perpetual occultation. Therefore, if ϕ represents the latitude of the observer, the ratio of the ever invisible portion of the celestial vault to the whole vault is manifestly $\sin^2\frac{1}{2}\phi$.

Also solved by A. H. Holmes, and the Proposer.

271. Proposed by W. J. GREENSTREET, M. A., Stroud, England.

Two equal concentric ellipses have their axes at an angle θ. Find the area of the quadrilateral circumscribing both, in terms of θ and the semi-axes.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denote the equation of one ellipse by $x^2/a^2+y^2/b^2=1$. The quadrilateral circumscribing both is evidently a rectangle whose sides are respectively parallel to the common chords. Their equations are, therefore,

$$x\cos\tfrac{1}{2}\theta+y\sin\tfrac{1}{2}\theta=P_1\,;\quad -x\sin\tfrac{1}{2}\theta+y\cos\tfrac{1}{2}\theta=P_2\,;\quad \text{where } P_2^2=a^2\cos^2\tfrac{1}{2}\theta+b^2\sin^2\tfrac{1}{2}\theta\,;$$
$$P_2^2=a^2\sin^2\tfrac{1}{2}\theta+b^2\cos^2\tfrac{1}{2}\theta.$$

The required area $=4P_1P_2=4\surd[(a^2-b^2)\sin^2\tfrac{1}{2}\theta\cos^2\tfrac{1}{2}\theta+a^2b^2]$.

Also solved by A. H. Holmes, A. S. Hawkesworth, and G. B. M. Zerr.

272. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

A point A revolves with uniform speed in a circle. A point B revolves around A, at a uniform distance from it, with the same angular velocity, but in the opposite direction. Determine the locus of B.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denote the center of the fixed circle by O, and take Ox through A and B at a time when these points are collinear, A lying between O and B. Let A move counter clockwise. Then $x=(a+b)\cos\theta$, $y=(a-b)\sin\theta$.

$\therefore \dfrac{x^2}{(a+b)^2}+\dfrac{y^2}{(a-b)^2}=1$, where a, b are the radii of the fixed and moving circles, respectively.

Also solved by A. H. Holmes, A. S. Hawkesworth, D. B. Northrup, G. B. M. Zerr, and Proposer.

GROUP THEORY.

11. Proposed by DR. SAUL EPSTEEN, University of Colorado, Boulder.

Find the six-parameter continuous group which leaves invariant the surface of second order $x_1x_2-x_3x_4=0$.

Solution by the PROPOSER.

To every transformation of this group defined as below, correspond four different sets of values of the parameters a, b, ... In particular to the identical transformation corresponds $a=d=\pm1$, $\alpha=\delta=\pm1$, $b=c=\beta=\gamma=0$.

The group is

$$x'_1=a(ax_1+bx_4)+\beta(ax_3+bx_2),$$
$$x'_2=\gamma(cx_1+dx_4)+\delta(cx_3+dx_2),$$
$$x'_3=\gamma(ax_1+bx_4)+\delta(ax_3+bx_2),$$
$$x'_4=a(cx_1+dx_4)+\beta(cx_3+dx_2).$$

MECHANICS.

181. Proposed by F. ANDEREGG, Professor of Mathematics, Oberlin College, Oberlin, Ohio.

A triangle AOB, of which the sides, OA, AB, and the angle at O are a, b, and α, revolves uniformly about O, so that OA makes the angle nt with the axis of x, and carries a circle of which AB is the diameter. Prove that a point moving in the circumference of the carried circle with twice the angular velocity of the triangle will describe an ellipse whose axes are

$$\sqrt{(a^2+b^2+2ab\cos\alpha)} \pm \sqrt{(a^2+b^2-2ab\cos\alpha)}.$$

Solution by F. H. SAFFORD, Ph. D., The University of Pennsylvania.

Let the third side of the the triangle be c and let the median from O be d, then

$$c=\tfrac{1}{2}\sqrt{(a^2+b^2-2ab\cos\alpha)}, \quad d=\tfrac{1}{2}\sqrt{(a^2+b^2+2ab\cos\alpha)}.$$

Consider the triangle at the time when the moving point is on the produced median and take the x-axis as coincident with the position of the median at this time. By taking the rotation of the moving point in the direction *opposite* to that of the triangle and noticing that the circle is itself subject to the rotation of the triangle on which it is fixed, the double velocity of the point causes it to have with respect to the axis of x a rotation opposite in direction and equal in amount to that of the triangle. Hence the coördinates of the moving point are

$$x=d\cos\theta+c\cos\theta,\ y=d\sin\theta-c\sin\theta, \text{ or } x=(d+c)\cos\theta,\ y=(d-c)\sin\theta,$$

from which follows the ellipse with axes as stated.

The problem may be executed mechanically as follows. At the point O fasten to the fixed plane a circle of twice the diameter of the carried circle. This circle is to remain fixed and to be connected with the carried circle by an *open* belt.

185. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A perfectly flexible rope whose weight is w per linear unit, and length $2l$, rests in equilibrium on a smooth peg. If now one end be raised a distance a and then released, find the time in which this end will rise to the height x above its original position, and the tension at that instant of the rope at the point where it passes over the peg.

Solution by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

Denote by z the height of one end above the initial position, and let m be the mass per unit of length. Then

$$2lm\frac{d^2z}{dt^2}=2mgz. \quad \therefore 2\frac{dz}{dt}\frac{d^2z}{dt^2}=\frac{2gz(dz/dt)}{l}.$$

$$\left(\frac{dz}{dt}\right)^2=\frac{g}{l}(z^2-a^2), \text{ since } z=a \text{ when } \frac{dz}{dt}=0.$$

$$\therefore t=\sqrt{\frac{l}{g}}\int\frac{dz}{\sqrt{(z^2-a^2)}}=\sqrt{\frac{l}{g}}\log\frac{z+\sqrt{(z^2-a^2)}}{a},$$

if $t=0$ when $z=a$. Putting $z=x$ we get the required time.

Also we have, [Tension at peg]δt=momentum generated by tension in time $\delta t=mv\delta t.v=mv^2\delta t$.

$$\therefore \text{Tension at peg}=m\left(\frac{dx}{dt}\right)^2=\frac{mg}{l}(x^2-a^2), \text{ when } z=x.$$

Also solved by S. A. Corey, and G. B. M. Zerr.

MISCELLANEOUS.

152. Proposed by J. EDWARD SANDERS, Reinersville, Ohio.

A conductor, the equation of the surface of which is

$$\frac{x^2}{25}+\frac{y^2}{16}+\frac{z^2}{9}=1,$$

is charged with 80 units of electricity, what is the density at a point for which $x=3$, $y=3$? If the density of this point be a, what is the whole charge on the ellipsoid? [From Peirce's *Potential Functions*, example 165, p. 388.]

Solution by M. E. GRABER, A. M., Tiffin, Ohio.

The mass of an ellipsoidal shell is $\frac{4}{3}\pi\rho d(abc)=4\pi Mabc$, and $Q=A4\pi\mu abc\rho$. $s=A\mu\theta$ and $\theta=\mu p$ where p is the perpendicular from the origin on the tangent plane. Then the density at any point is $Qp/4\pi abc$. The value of z for a point on the ellipsoid for which $x=3$ and $y=3$, is $\frac{3\sqrt{31}}{20}$ and the equation of the tangent plane is $\frac{3x}{25}+\frac{3y}{16}+\frac{3\sqrt{31}}{9}z=1$. The perpendicular from (0, 0, 0) is $\frac{D}{\sqrt{(A^2+B^2+C^2)}}$ or 4.2+. Therefore the density at $\left(3, 3, \frac{3\sqrt{21}}{20}\right)=\frac{80.(4.2)}{4\pi(5.4.3)}$ $=.445+$. If the density of this part be a,

$$Q=\frac{4\pi a(5.4.3)}{4.2+}=\frac{240\pi a}{4.2+}.$$

Also solved by G. B. M. Zerr, and the Proposer.

153. Proposed by CHRISTIAN HORNUNG, A. M., Heidelberg University, Tiffin, Ohio.

Two men start from Columbus, Ohio, at the same time; one travels east and the other west. They travel at the rate of 4 miles an hour from sunrise to sunset each day until they meet. Where will they meet and what distance will each have traveled?

Solution by A. H. HOLMES, Brunswick, Me.

We assume that Columbus, Ohio, is situated at 40° N. latitude, and 83° W. longitude. $2\pi x$=distance round the earth at 40° latitude. Put $a=\frac{1}{2}$ equatorial diameter of the earth, and $b=\frac{1}{2}$ polar diameter of the earth.

Then $a^2y^2+b^2x^2=a^2b^2$. $y=x\tan 40°$.

$\therefore 3962.8^2(.704088x^2)+3949.57^2x^2=3962.8^2\times 3949.57^2$. $\therefore x=3031.48$.

Then A going east will have travelled when he meets B, $\frac{8\pi x-384}{16\pi x}\times 2\pi x=\pi x-48$; and B, $\pi x+48$ miles.

$\therefore$ A travelled 9475.6899+ miles, B travelled 9571.6899+ miles. A will have passed 179° 5′ of longitude.*

Hence they will meet at a point in the Eastern Desert of the Chinese Empire situated at 40° N. latitude, and 96° 5′ E. longitude.

Also solved by L. E. Newcomb, J. Scheffer, and G. B. M. Zerr.

154. Proposed by D. BIDDLE (Unsolved problem in the Educational Times, London).

Prove that the proper angle at which to cross a street when a person wishes to continue his course on the other side, and the roadway is n times as muddy as the pavement, is that of which the sine is $(n^2-1)/(n^2+1)$.

Solution by G. B. M. ZERR, A. M., Ph. D., Parsons, W. Va.

Since the roadway is n times as muddy as the pavement, the velocity in the roadway is $1/n$th the velocity on the pavement.

Let A be the position of the person, B the point directly opposite him on the other side of the roadway, C the point where he reaches the other side, $AB=a$, $\angle CAB=\frac{1}{2}\pi-\theta$. Also let v be his velocity on the pavement. Then to get Mr. Biddle's result, the time to go over AB plus the time to go over BC, or

$$\frac{n.AB}{v}+\frac{BC}{v}=\frac{n.AC}{v}. \qquad \therefore \frac{na}{v}+\frac{a\cot\theta}{v}=\frac{na\operatorname{cosec}\theta}{v}.$$

$$\therefore n\sin\theta+\cos\theta=n. \qquad \therefore \sin\theta=\frac{n^2-1}{n^2+1}.$$

I prefer the following result: Let D be the person's goal, $BD=b$, $AC=a\operatorname{cosec}\theta$, $BC=a\cot\theta$.

Then $\frac{na\operatorname{cosec}\theta}{v}+\frac{b-a\cot\theta}{v}=$minimum.

$\therefore n\operatorname{cosec}\theta\cot\theta=\operatorname{cosec}^2\theta$ or $\cos\theta=1/n$. $\therefore \sin\theta=\sqrt{(n^2-1)}/n$.

Also solved by A. H. Holmes, and J. Scheffer.

*Mr. Newcomb in his solution used more exact assumptions in regard to the latitude of Columbus, and tabulated the data showing the progress of A and B on successive days. The result of his long solution is 179° 33′ 13.1″ East of Columbus. G.

PROBLEMS FOR SOLUTION.

ALGEBRA.

250. Proposed by PROFESSOR WILLIAM HOOVER, Ph. D., Athens, Ohio.

Factor $a^2b^2(x^2+y^2)(a^2y^2+b^2x^2-a^2b^2)=(a^4y^2+b^4x^2)[\sqrt{(a^2y^2+b^2x^2)}+ab]^2$.

251. Proposed by S. A. COREY, Hiteman, Iowa.

Prove that $\frac{1}{n+1}+\frac{1}{2(n+2)}+\frac{1}{3(n+3)}+$ etc.,=

$$\frac{1}{n^2}+\tfrac{1}{2}\left[\frac{1}{n-1}+\frac{1}{2(n-2)}+\frac{1}{3(n-3)}+\ldots+\frac{1}{l(n-l)}\right],$$

l being equal to $(n-1)$, n being any positive integer greater than one.

252. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

Solve (1) $x-y=\frac{1}{3}\pi$; (2) $\sin x=\cos^3 y$.

AVERAGE AND PROBABILITY.

174. Proposed by HENRY HEATON, Atlantic, Iowa.

Chords are drawn through every point of the surface of a given circle in every possible direction. What is their average length?

175. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

If a line l is divided into three parts by two points taken at random on it, what is the mean value of the triangle whose sides are equal to the three parts? (Only those cases are to be considered in which the the three parts will form a triangle.)

CALCULUS.

212.* Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

Show that any root of the equation $y^5-5y=4x$ satisfies the differential equation $\frac{d^4y}{dx^4}=(\frac{4}{5})^4 x^{-3}\frac{d^4(x^{-\frac{1}{4}}y)}{d(x^{-\frac{1}{4}})^4}$. Generalize the problem.

213. Proposed by EDWIN L. RICH, Schenectady, N. Y.

Let $f(x)$ be any function of x, and $f'(x)$ its derivative. If $u=[f'(x)]^{-\frac{1}{2}}$, $v=f(x)[f'(x)]^{-\frac{1}{2}}$, then $\frac{1}{u}\frac{d^2u}{dx^2}-\frac{1}{v}\frac{d^2v}{dx^2}=0$.

*The problem is due to Heymann.

214. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

Prove that $\pi^2 = 6.\frac{2^2}{2^2-1}.\frac{3^2}{3^2-1}.\frac{5^2}{5^2-1}$ ..... where the squared numbers in the numerator are the natural *primes* in order.

GEOMETRY.

277. Proposed by G. W. GREENWOOD, M. A., McKendree College, Lebanon, Ill.

It is tacitly assumed in elementary geometry that as the number of sides of a regular polygon inscribed in a circle is increased, *in any manner*, that its perimeter has a fixed limit. Beginning with a square and then continually doubling the number of sides we get for the perimeter $2^{n+2}\sqrt{[2-E^n(0)]}$, where $E(x) \equiv \sqrt{(2+x)}$. Beginning with a hexagon we get $2^{m+1}3\sqrt{[2-E^m(1)]}$. The definition of the length of a circle assumes that these expressions have the same limit as $n \doteq \infty$ and $m \doteq \infty$. Prove it.

278. Proposed by L. E. NEWCOMB, Los Gatos, Cal.

AF, MN are parallel lines indefinitely extended toward FN; at right angles to AF, MN is AM of length 22; upon the base AB, which is in line with AM, is the triangle ABC whose sides are $AB=21$, $BC=10$, $AC=17$; find the sides of the maximum similar triangle with base extending from B to some point in AF, the vertex in line with MN.

279. Proposed by C. C. WENTWORTH, C. E., Roanoke, Va.

To construct geometrically the maximum equilateral triangle circumscribed about a given triangle.

GROUP THEORY.

12. Proposed by GEORGE H. HALLETT, Ph. D., Assistant Professor of Mathematics, The University of Pennsylvania.

*Given $U_1=\alpha'$, $V_1=\beta'$, and the recursion formulae $U_y=\alpha' V_{y-1}+\alpha'' U_{y-1}$, $V_y=\beta' V_{y-1}+\beta'' U_{y-1}$. Find expressions for U_y, V_y in terms of the coefficients α', α'', β', β''.

13. Proposed by O. E. GLENN, Ph. D., Springfield, Mo.

The order of the linear homogeneous group in n letters is $(p^n-1)(p^n-p)$ .....(p^n-p^{n-1}). Two proofs are given in Burnside's *Finite Groups*. Give other proofs.

*The problem is of frequent occurrence in abstract group construction.

MISCELLANEOUS.

155. Proposed by A. H. HOLMES, Brunswick, Maine.

There are two vessels, one containing a gallons of alcohol, the other b gallons of water. Suppose that c gallons are simultaneously taken from each and poured into the other, how many times must this be done so that there will be the same proportion of alcohol to water in each vessel?

156. Proposed by R. D. CARMICHAEL, Hartselle, Ala.

There exist no multiply perfect odd numbers of multiplicity n containing only n distinct primes.

UNSOLVED PROBLEMS.

NOTE. The following problems still remain unsolved (in our columns).

Average and Probability, 164 Proposed by J. O. MAHONEY, B.E., M.Sc., Central High School, Dallas, Tex.

If m is prime, and the numbers 0, 1, 2, ..., m^2-1 are placed at random in the form of a square, the probability that the square is hyper-magic is $(m-1)m/(m^2-2)$!

Algebra, 179. Proposed by DR. L. E. DICKSON, The University of Chicago.

Find the roots of the algebraically solvable quintic equation

$$x^5+qx^2+px+\tfrac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0.$$

NOTES AND NEWS.

Mr. B. M. Rastall is assistant in mathematics at the University of Wisconsin.

Mr. E. A. Moritz, Mr. R. S. Peatter, and Mr. E. R. Smith have been appointed instructors in mathematics at the University of Wisconsin.

The University of North Carolina has granted a years leave of absence to Mr. M. H. Stacy, who will pursue a graduate course in mathematics at Cornell University.

The American Association for the Advancement of Science held its annual meeting during the holidays at New Orleans, La. Section A: Mathematics and Astronomy, was presided over by Dr. W. S. Eichelberger, of the United States Naval Observatory at Washington, D. C. Professor L. G. Weld, of the University of Iowa, is secretary of the Section.

The graduate department of the University of Cincinnati has been reorganized with the title of graduate school. Its faculty consists of the heads of departments of the University.

The Mathematical Club of the University of Washington was organized this year, with Professor Moritz, president, F. M. Morrison, secretary. The meetings of the Club are monthly and are devoted to reports on original work, and the reviews of current mathematical literature.

A comparison of the membership of the world's mathematical societies of national character places Germany first and the United States second. The Deutsche Mathematiker Vereinigung has 666 members, while the membership of the American Mathematical Society is slightly over 500. The London Mathematical Society has between 290 and 300 members.

John Henry Serviss, civil engineer and surveyor, died at his residence in Closter, Bergen County, New Jersey, on August 26th. Mr. Serviss was born in the town of Glen, Montgomery County, New York, August 17th, 1837, and was graduated from Union College in 1863. Since 1868, he has practiced his profession as civil engineer throughout northern New Jersey with distinguished ability.

During the past year there was created at the University of Washington a new assistant professorship of mathematics which was filled by the appointment of Professor F. M. Morrison. Over $800 worth of new books have been added to the mathematical library during the past year and $200 worth of models. This progress in the department of mathematics is in keeping with the general growth of that university.

The following have been elected members of the American Mathematical Society: Professor O. P. Akers, Allegheny College; Dr. R. B. Allen, Clark University; Professor Ernesto Cesàro, University of Naples; Lieutenant Colonel A. J. C. Cunningham, London, Eng.; Miss M. E. Decherd, University of Texas; Mr. W. W. Hart, Shortridge High School, Indianapolis, Ind.; Mr. H. N. Olsen, Bethany College; Mr. F. H. Smith, Southwestern Christian College.

The Missouri Society of Teachers of Mathematics held its regular Winter meeting at Jefferson City on December 27 and 28. The following papers were read: Maxima and Minima, by George R. Dean, Missouri School of Mines; Laboratory Methods in Algebra Teaching, by Oliver E. Glenn, Drury College; The Treatmeht of Limits in Elementary Geometry, by Albert M. Wilson, McKinley High School, St. Louis; Some Problems in Arithmetic in the Grades, by Thos. P. Jandon, Kansas City. The Thursday afternoon program was devoted to a round table discussion, "what should be taught in arithmetic and what omitted."

We learn from *Science* that the annual meeting of the Association of Teachers of Mathematics in the Middle States and Maryland was held on Saturday, December 2, in affiliation with the Association of Colleges and Preparatory Schools of the Middle States and Maryland. The following papers were presented:

Professor H. S. White, Vassar College, How Should the College Teach Analytic Geometry. Mr. H. R. Higly, Pennsylvania College: Suggestions for the First Twelve Lessons in Demonstrative Geometry. Dr. John S. French, Jacob Tome Institute, Port Deposit: Some Essentials of the Successful Mathematics Teacher. Dr. H. A. Converse, Baltimore Polytechnic Institute: The Teaching of Geometry.

The association was disappointed at not being able to listen to a paper on The Teaching of Pure and Applied Mathematics, which the program announced was to be read by President R. S. Woodward, of Carnegie Institute, Washington.

The following officers were elected for the coming year: President, Professor Edwin S. Crawley, University of Pennsylvania. Vice President, Dr. John S. French, Jacob Tome Institute, Port Deposit, Md. Secretary and Treasurer, Dr. J. T. Rorer, Central High School, Philadelphia, Pa. Members of the Council, Professor W. H. Metzler, Syracuse University; Miss L. G. Simons, New York City Normal College; Dr. J. L. Patterson, Chestnut Hill Academy, Philadelphia, Pa.; Professor W. H. Maltbie, Woman's College of Baltimore.

At the meeting the following resolution was adopted:

Resolved, That this association approve of the organization of a national federation of existing associations of teachers of mathematics in which each association shall preserve its own organization and individuality and which shall have among its objects the joint support of a publication. In the federation should be included only societies representing territory as extensive at least as one State.

The Association of Ohio Teachers of Mathematics and Science holds its third annual meeting in Townsend Hall, Ohio State University, Columbus, on December 28. The following papers will be read before the mathematical section: Do the mathematical courses in literary colleges property fit for the mathematics of engineering?—Christian Hornung, Tiffin. Recent contributions marking a real advance in the theory of mathematics teaching.—Anna H. Palmie, Cleveland. Sir Isaac Newton—an estimate.—C. L. Arnold, Columbus. A contribution from non-Euclidean geometry to school spherics.—George Bruce Halsted. Symbolism in mathematics.—Harriet E. Glazier, Oxford. The character of mathematics teaching in the high school.—W. T. Heilman, Columbus. Report of the committee on "A straight line is the shortest distance or path between two points.—George Bruce Halsted, Gambier. Preliminary report of the committee on a syllabus of the fundamental propositions of elementary geometry proposed for consideration by the Association of Mathematical Teachers in New England. —George Bruce Halsted, Gambier. The influence of college entrance certificates on the teaching of mathematics in the high school.—Harry E. Giles, Kenton.

BOOKS AND PERIODICALS.

Theory of Functions of Real Variables. By James Pierpont, Professor of Mathematics in Yale University. Vol. I. 8vo. Cloth, xii+560 pages. Price, $4.50. Boston: Ginn & Co.

This work, which the author says is based upon a course of lectures given by him at Yale University, is designed to present the more important results in the theory of functions of real variables. Vol. I is devoted to the foundations of the Differential and Integral Calculus. It is divided into sixteen chapters. In Chapter I, rational numbers, the laws governing their generation, and some of their properties are discussed; Chapter II, deals in the same way with irrational numbers; Chapter III, with exponentials and logarithms; Chapter IV, with elementary functions; Chapter V, with first notions of point aggregates; Chapter VI, with limits of functions; Chapter VII, with continuity and discontinuity of functions; Chapter VIII, differentiation; Chapter IX, with implicit functions; Chapter X, with indeterminate forms; Chapter XI, with maxima and minima; Chapter XII, with integration; Chapter XIII, with proper integrals; Chapter IV, improper integrals, integrand infinite; Chapter XV, improper integrals, interval of integration infinite; and Chapter XVI, multiple proper integrals.

The theorems are first explicitly stated and then proved. Many examples of incorrect forms of reasoning currently found in standard works on the calculus are given to emphasize the necessity of a more critical study of fundamentals and at the same time "to stimulate the critical sense of the student." While that part of the work developing the theory of irrationals as introduced by Cantor and Dedekind, the theory of point aggregates or masses of points, and the theory of discontinuous functions are not as fully treated as those familiar with the profound researches of Cantor, Weierstrass, Neumann, and others might wish, yet the author for writing and the publishers for publishing it, have placed the American mathematical world under obligations to them. It is extremely gratifying to see an American publishing company taking upon itself the responsibility of publishing such works as this and Goursat's *Cours d'Analyses*, with no possibility of great financial returns, but solely with a view of encouraging the development of mathematics in America. They should receive, therefore, every possible encouragement for the interest they are thus manifesting in disseminating by such publications, the refined views of modern mathematicians. B. F. F.

A Course in Mathematical Analysis. By Édvard Goursat, Professeur à la Faculté des Sciences de Paris. Authorized translation by E. R. Hedrick, Professor of Mathematics in the University of Missouri. Vol. I. 8vo. Cloth, viii+548 pages. Illustrated with 52 diagrams. Price, $4.00. Boston: Ginn & Co.

The French edition of this valuable work appeared in 1902, and a critical review of it, by Professor W. F. Osgood, of Harvard University, appeared in the *Bulletin of the American Mathematical Society*, Vol. IX, No. 10, 1903. At Professor Osgood's suggestion, the translation of the work into English was undertaken.

In this work, the fundamental principles of the calculus are established with great rigor and thoroughness. It begins with the foundations of the calculus, assuming, however, that the student has already had an elementary course in this subject. In the first chapter, precise definitions of *limit*, *function*, *derivative*, etc., are given. Chapter II is an exposition of the modern treatment of implicit functions, functional relations, and transformations. In Chapter III, Taylor's Theorem and its applications are presented in an admirable manner. Also the subject of maxima and minima is treated with equal rigor and elegance. The ambiguous case is treated with considerable fullness. The word *extremum* is used as a generic term for maximum and minimum. Chapters IV, V, VI, and VII deal

with definite and indefinite integrals from the modern point of view. The definite integral is defined as sum and the fundamental properties and formulas are established from it. In Chapter VIII, infinite series, including series of imaginary terms, and the theory of uniform convergence are discussed. Chapter IX supplements Chapter VIII with a treatment of power series and trigonometric series, including Fourier's Series. Chapters X, XI, and XII deal with curves and surfaces.

The work is one that is sure to appeal strongly to those teachers of mathematics who wish to put into the hands of their students a book eminently scholarly and accurate in treatment and embracing the most modern materials and applications. An equal service would be rendered to the American mathematical public, were some one to put into good English with the necessary annotations and proofs of many of the unproved theorems of Dini's *Theory of Functions*, and other epoch-making works on mathematics which are generally unknown to most teachers of mathematics. B. F. F.

The current number of the *Proceedings of the London Mathematical Society* contains the following papers:

The Intersection of Two Conic Sections (continued), by Mr. J. A. H. Johnston; On the Projection of Two Triangles on the Same Triangle, by Professor M. J. M. Hill, Dr. L. N. G. Filon, and Mr. H. W. Chapman; On the Condition of Reducibility of any Group of Linear Substitutions, by Professor W. Burnside; On a Class of Analytic Functions, by Mr. G. H. Hardy; Linear Content of a Plane Set of Points, by Dr. W. H. Young.

ERRATA.

On page 207, line 5 from bottom, for $\frac{d^2\theta'}{dt'}$ read $\frac{d^2\theta'}{dt'^2}$.

On page 208, line 6 trom bottom, for $\frac{d^2\theta''}{dt''_2}$ read $\frac{d^2\theta''}{dt''^2}$.

On page 211, line 22 should read, $x^5+qx^2+px+\frac{1}{5}\left[\frac{q^2}{p}-\frac{p^3}{5q}\right]=0$.

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ARITHMETIC. Moore and Miner's *Practical Business Arithmetic*, p. 198.

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